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EXPLICIT SOLUTION OF THE INVERSE PROBLEM FOR A VIBRATING STRING--ETC(U)
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The problem of reconstructing the density $\rho(x)$ of a vibrating string of length L from the knowledge of two spectra $\{\lambda_n\}_1^\infty$ and $\{\mu_n\}_1^\infty$ is considered. The method of construction relies (i) on a formula for ρ at an arbitrary point $x=\ell$ in terms of the spectra $\{\lambda_n(\ell)\}_1^\infty$ and $\{\mu_n(\ell)\}_1^\infty$ associated with the natural

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20. Abstract (Continued)

frequencies of vibration of the portion (ℓ,L) of the original string and (ii) on a set of first order differential equations for $\{\lambda_n(\ell)\}_1^\infty$ and $\{\mu_n(\ell)\}_1^\infty$. The density is deduced by integrating these equations of the spectra and substituting in the above mentioned formula.

Explicit Solution of the Inverse Problem for a Vibrating String

by

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1. Introduction

The vibrations of a string of density $\rho(x)$, length L taut by a unit tension, are governed by the equation

$$u'' + \omega^2 \rho u = 0, \quad x \in (0,L). \tag{1}$$

It is well known that if $\rho \in C^2(0,L)$, this equation can be transformed into the canonical Sturm-Liouville equation

$$y'' + (\omega^2 - q)y = 0.$$

However, even in those cases in which ρ is sufficiently smooth, there are certain advantages in dealing with the problem in its original form (1), particularly if one is interested in other vibrating problems.

In the present paper, I shall present a new approach to the solution of the inverse problem for the vibrating string, i.e. to the reconstruction of the density $\rho(x)$ given its length L and two spectra $\{\lambda_n\}_1^\infty$ and $\{\mu_n\}_1^\infty$. In order to simplify the presentation, I shall assume that the spectrum $\{\lambda_n\}_1^\infty$ is associated with (1) and the boundary conditions

$$u(0) = u(L) = 0, \tag{2}$$

i.e. the fixed/fixed configuration, while $\{\mu_n\}_1^\infty$ is associated with the free/fixed configuration:

$$u'(0) = u(L) = 0. \tag{3}$$

The data $\{\lambda_n\}_1^\infty$, $\{\mu_n\}_1^\infty$ and L guarantee the uniqueness of the solution $\rho(x)$. The existence of this solution is also guaranteed if the eigenvalues μ_n and λ_n (i) interlace, (ii) have the appropriate asymptotic behavior for large order and (iii) are such that [1]

$$\sum_{n=1} \frac{1}{\mu_n^2 \prod_{k\neq n}^{\infty} (1-\frac{\mu_n^2}{\mu_k^2}) \prod_{k=1}^{\infty} (1-\frac{\mu_n^2}{\lambda_k^2})} < \infty .$$

In addition to the above conditions, I shall assume that the data are such that $\rho(x)$ is continuous and bounded away from zero.

2. Statement of Results

<u>Theorem 1</u>: Let $\{\lambda_n\}_1^\infty$ be the spectrum of the eigenvalue problem

$$\left.\begin{array}{l} u'' + \omega^2 \rho u = 0 , \\ u(0) = u(L) = 0 , \end{array}\right\}$$

and $\{\mu_n\}_1^\infty$ the spectrum of

$$\left.\begin{array}{l} u'' + \omega^2 \rho u = 0 , \\ u'(0) = u(L) = 0 . \end{array}\right\}$$

If the function $\rho(x)$ is continuous and bounded away from zero, then

$$\rho(0) = \frac{1}{L^2 \mu_1^2} \prod_{n=1}^{\infty} \frac{\lambda_n^4}{\mu_{n+1}^2 \mu_n^2} .$$

<u>Corollary</u>: Let $\{\lambda_n(\ell)\}_1^\infty$ be the spectrum of the eigenvalue problem

$$\left.\begin{aligned} &u'' + \omega^2 \rho u = 0 \ , \\ &u(\ell) = u(L) = 0 \ , \end{aligned}\right\}$$

and $\{\mu_n(\ell)\}_1^\infty$ the spectrum of

$$\left.\begin{aligned} &u'' + \omega^2 \rho u = 0 \ , \\ &u'(\ell) = u(L) = 0 \ , \end{aligned}\right\}$$

where $0 \leq \ell < L$. If $\rho(x)$ is continuous and bounded away from zero for $x \varepsilon (0,L)$, then

$$\rho(x) = \frac{1}{(L-x)^2 \ \mu_1^2(x)} \prod_{n=1}^{\infty} \frac{\lambda_n^4(x)}{\mu_{n+1}^2(x) \mu_n^2(x)} .$$

<u>Theorem 2</u>: Let $\{\lambda_n\}_1^\infty$, $\{\mu_n\}_1^\infty$ and L be such that the solution $\rho(x)$ of the inverse problem for the vibrating string exists and is continuous and bounded away from zero. Then, this solution is given by

$$\rho(x) = \frac{1}{(L-x)^2 \ \mu_1^2(x)} \prod_{n=1}^{\infty} \frac{\lambda_n^4(x)}{\mu_{n+1}^2(x) \ \mu_n^2(x)}$$

where $\{\lambda_n(x)\}_1^\infty$ and $\{\mu_n(x)\}_1^\infty$ are the solutions of the initial value problems

$$\frac{d\lambda_n^2}{dx} = - \frac{\lambda_n^2(x)}{L-x} \cdot \frac{\prod_{k=1}^{\infty} (1-\lambda_n^2(x)/\mu_k^2(x))}{\prod_{k \neq n}^{\infty} (1-\lambda_n^2(x)/\lambda_k^2(x))} ,$$

$$\frac{d\mu_n^2}{dx} = \mu_n^4(x)\rho(x)(L-x)\,\frac{\prod_{k=1}^{\infty}(1-\mu_n^2(x)/\lambda_k^2(x))}{\prod_{k\neq n}^{\infty}(1-\mu_n^2(x)/\mu_k^2(x))}\,,$$

with

$$\lambda_n^2(0) = \lambda_n^2\,,$$

$$\mu_n^2(0) = \mu_n^2\,.$$

3. Proofs

Proof of Theorem 1. The given data are such as to insure the existence and uniqueness of the solution of the inverse problem. Following Krein [1,2], we can construct this solution by considering a converging sequence of approximations to the integrated mass, viz.

$$m(x) = \int_0^x \rho(x')dx'\,.$$

The generic Nth order approximation to m(x), say $m^{(N)}(x)$, is a piece-wise constant function:

$$m^{(N)}(x) = \sum_{j=0}^{i} m_j \quad \text{for} \quad x\varepsilon[x_i,\, x_{i+1}), \quad i=1,2,\ldots,N \tag{4}$$

In the above formula m_o=0, x_o=0 and x_{N+1}=L. The other values of m_i and x_i, or rather

$$\ell_i = x_{i+1} - x_i ,$$

are found by writing the rational fraction $-U_o(\omega^2)/U_o'(\omega^2)$ where

$$U_o(\omega^2) = -L \prod_{n=1}^{N} (1 - \omega^2/\lambda_n^2) , \tag{5a}$$

and

$$U_o'(\omega^2) = \prod_{n=1}^{N} (1 - \omega^2/\mu_n^2) , \tag{5b}$$

as a Stieltjes continued fraction, viz.

$$-\frac{U_o(\omega^2)}{U_o'(\omega^2)} = \ell_o + \frac{1|}{|-m_1\omega^2} + \frac{1|}{|\ell_1} + \ldots + \frac{1|}{|\ell_N} . \tag{6}$$

It is possible to write down explicit formulas for ℓ_o and m_1. To that effect, we first divide $-U_o(\omega^2)$ by $U_o'(\omega^2)$, viz.

$$-U_o(\omega^2) = \ell_o U_o'(\omega^2) - U_1(\omega^2) , \tag{7}$$

and then $U_o'(\omega^2)$ by the remainder of the previous division, namely $-U_1(\omega^2)$:

$$U_o'(\omega^2) = -m_1\omega^2 U_1(\omega^2) + U_1'(\omega^2) . \tag{8}$$

Clearly

$$\ell_o = L \prod_{n=1}^{N} \frac{\mu_n^2}{\lambda_n^2} \quad . \tag{9}$$

The polynomial $U_1(\omega^2)$ being of degree (N-1) can be written as follows:

$$U_1(\omega^2) = -(L-\ell_o) \prod_{n=1}^{N-1} \left(1- \frac{\omega^2}{\tilde{\lambda}_n^2}\right) \quad . \tag{10}$$

where $\tilde{\lambda}_n$ is the nth eigenfrequency of a string with point masses $\{m_i\}_2^N$ located at $\{x_i\}_2^N$ whose left end at $x=x_1\equiv\ell_o$ and right end at $x=L$ are held fixed. With this expression for $U_1(\omega^2)$ we can carry out the division in (8) and get

$$m_1 = \frac{1}{(L-x_1)} \cdot \frac{\prod_{n=1}^{N-1} \tilde{\lambda}_n^2}{\prod_{n=1}^{N} \mu_n^2} \quad . \tag{11}$$

Now consider the ratio m_1/ℓ_o: by definition

$$\frac{m_1}{\ell_o} = \frac{\int_o^{x_1} \rho^{(N)}(x)dx}{x_1}$$

where $\rho^{(N)}(x)$ is the N-th approximation to the density. If ρ is bounded away from zero, then as $N\to\infty$, $x_1\to 0$ and the above ratio tends to $\rho(0)$. Therefore

$$\rho(0) = \lim_{N\to\infty} \frac{1}{(L-x_1)L\,\mu_1^2} \cdot \prod_{n=1}^{N-1} \frac{\lambda_n^2\,\tilde{\lambda}_n^2}{\mu_{n+1}^2\,\mu_n^2} \cdot \frac{\lambda_N^2}{\mu_N^2} \ .$$

As $N\to\infty$, $x_1\to 0$ and consequently $\tilde{\lambda}_n\to\lambda_n$; making use of the asymptotic form of the eigenvalues, we can show that the resulting infinite product is convergent and consequently that

$$\rho(0) = \frac{1}{L^2\mu_1^2} \prod_{n=1}^{\infty} \frac{\lambda_n^4}{\mu_{n+1}^2\,\mu_n^2} \ . \tag{12}$$

By repeating the analysis for that portion of the string occupying the segment (ℓ,L) we could deduce that

$$\rho(\ell) = \frac{1}{(L-\ell)^2\,\mu_1^2(\ell)} \prod_{n=1}^{\infty} \frac{\lambda_n^4(\ell)}{\mu_{n+1}^2(\ell)\mu_n^2(\ell)} \ . \tag{13}$$

<u>Proof of Theorem 2.</u> Let $U(x,\omega^2)$ denote a fundamental solution of (1) such that

$$U(L,\omega^2) = 0 \ ,$$

$$U'(L,\omega^2) = 1 \ .$$

Since $U(x,\omega^2)$ and $U'(x,\omega^2)$ are entire functions of ω^2 of order 1/2, they can be written as follows:

$$\left.\begin{aligned} U(x,\omega^2) &= -(L-x)\prod_{n=1}^{\infty}(1-\omega^2/\lambda_n^2(x)) \quad , \\ U'(x,\omega^2) &= \prod_{n=1}^{\infty}(1-\omega^2/\mu_n^2(x)) \quad . \end{aligned}\right\} \tag{14}$$

The coefficients in front of the infinite products have been determined by setting ω^2 equal to zero in (1). The zeros of $U(x,\omega^2)$ and $U'(x,\omega^2)$ are the eigenvalues of that portion of the string occupying the interval (x,L) and vibrating in the fixed/fixed and free/fixed configurations.

The product representations (14) must (i) be compatible and (ii) satisfy the original equation (1). As a result, we must have

$$\prod_{k=1}^{\infty}(1-\omega^2/\lambda_k^2(x)) - (L-x)\sum_{n=1}^{\infty}\frac{\omega^2}{\lambda_n^4(x)}\frac{d\lambda_n^2}{dx}\prod_{k\neq n}^{\infty}\left(1-\frac{\omega^2}{\lambda_k^2(x)}\right)$$

$$= \prod_{k=1}^{\infty}\left(1-\frac{\omega^2}{\mu_k^2(x)}\right) \quad ,$$

and

$$\sum_{n=1}^{\infty}\frac{\omega^2}{\mu_n^4(x)}\frac{d\mu_n^2}{dx}\prod_{k\neq n}^{\infty}\left(1-\frac{\omega^2}{\mu_k^2(x)}\right)$$

$$= \omega^2\rho(x)(L-x)\prod_{k=1}^{\infty}\left(1-\frac{\omega^2}{\lambda_k^2(x)}\right) \quad .$$

By equating the coefficients of equal powers of ω^2, the above identities give rise to an infinite system of linear equations for the unknowns $\{d\lambda_n^2/dx\}_1^\infty$ and $\{d\mu_n^2/dx\}_1^\infty$. The solution of this infinite system can be obtained via the theory of infinite determinants. However, there is a simpler way of arriving at the solution. To that effect we return to (14) and write

$$U(x, \lambda_n^2(x)) = 0 , \quad n=1, 2, \ldots$$

Differentiating with respect to x, we get:

$$\left.\frac{\partial U(x,\omega^2)}{\partial x}\right|_{\omega^2 = \lambda_n^2(x)} + \frac{d\lambda_n^2}{dx} \left.\frac{\partial U(x,\omega^2)}{\partial \omega^2}\right|_{\omega^2 = \lambda_n^2(x)} = 0$$

and making use of the product representations (14), we deduce that

$$\frac{d\lambda_n^2}{dx} = - \frac{\lambda_n^2(x)}{L-x} \cdot \frac{\prod_{k=1}^{\infty} \left(1 - \frac{\lambda_n^2(x)}{\mu_k^2(x)}\right)}{\prod_{k\neq n}^{\infty} \left(1 - \frac{\lambda_n^2(x)}{\lambda_k^2(x)}\right)} . \tag{15}$$

Similarly

$$\frac{d\mu_n^2}{dx} = - \mu_n^4(x) \, \rho(x) (L-x) \frac{\prod_{k=1}^{\infty} \left(1 - \frac{\mu_n^2(x)}{\lambda_k^2(x)}\right)}{\prod_{k\neq n}^{\infty} \left(1 - \frac{\mu_n^2(x)}{\mu_k^2(x)}\right)} . \tag{16}$$

Or, substituting $\rho(x)$ by the expression given in (13),

$$\frac{d\mu_n^2}{dx} = \frac{1}{L-x} \cdot \frac{\mu_n^4(x)}{\mu_1^2(x)} \prod_{m=1}^{\infty} \frac{\lambda_m^4(x)}{\mu_{m+1}^2(x)\mu_m^2(x)} \cdot \frac{\prod_{k=1}^{\infty} (1-\mu_n^2(x)/\lambda_k^2(x))}{\prod_{k\neq n}^{\infty} (1-\mu_n^2(x)/\mu_k^2(x))} \ . \tag{17}$$

We shall establish next that the initial value problem consisting of (15),(17) subject to the initial conditions

$$\begin{aligned} \lambda_n^2(0) &= \lambda_n^2 \ , \\ & \qquad\qquad (n=1,\ 2,\ \ldots) \\ \mu_n^2(0) &= \mu_n^2 \ , \end{aligned} \tag{18}$$

has a unique solution. For convenience, we can write this initial value problem as

$$\frac{d\nu}{d\xi} = h(\nu) \quad ,$$

$$\nu(0) = \nu \quad ,$$

where the stretched coordinate ξ is defined thus

$$\xi = \ell n \ (1 - \frac{x}{L}) \ ,$$

ν stands for the sequence obtained by interlacing the eigenvalues $\{\mu_n^2\}_1^\infty$ and $\{\lambda_n^2\}_1^\infty$, namely

$$\nu_{2n} = \lambda_n^2 , \qquad (n=1,\ 2,\ \ldots)$$
$$\nu_{2n-1} = \mu_n^2 ,$$

and finally

$$h_{2n} = f_n \equiv -\lambda_n^2 \frac{\prod_{k=1}^{\infty} (1-\lambda_n^2/\mu_k^2)}{\prod_{k\neq n}^{\infty} (1-\lambda_n^2/\lambda_k^2)} , \tag{19a}$$

$$h_{2n-1} = g_n \equiv \frac{\mu_n^4}{\mu_1^2} \prod_{m=1}^{\infty} \frac{\mu_m^4}{\mu_{m+1}^2 \mu_m^2} \cdot \frac{\prod_{k=1}^{\infty} (1-\mu_n^2/\lambda_k^2)}{\prod_{k\neq n}^{\infty} (1-\mu_n^2/\mu_k^2)} , \tag{19b}$$

Let us first prove that f_n and g_n are of $O(n^2)$ for a given pair of spectra $\{\mu_n\}_1^\infty$, $\{\lambda_n\}_1^\infty$. To that effect we write f_n and g_n in a more convenient form:

$$f_n = n(\lambda_n^2-\mu_n^2) \cdot \prod_m \frac{\lambda_m^2}{\mu_m^2} e^{-1/m} \cdot e^{\frac{1}{n} - \ell nn} \prod_{k\neq n} \frac{\mu_k^2-\lambda_n^2}{\lambda_k^2-\lambda_n^2} e^{1/k} \tag{20a}$$

and

$$g_n = n(\lambda_n^2-\mu_n^2) \cdot \prod_m \frac{\lambda_m^2}{\lambda_{m+1}^2} e^{1/m} \cdot e^{-\frac{1}{n} - \ell nn} \frac{\mu_n^2}{\mu_1^2} \prod_{k\neq n} \frac{\lambda_k^2-\mu_n^2}{\mu_k^2-\mu_n^2} e^{-1/k} . \tag{20b}$$

Since

$$\left.\begin{aligned} \lambda_n^{\,2} &\sim n^2\pi^2/\left(\int_o^L \rho^{1/2}dt\right)^2 \\ \mu_n^{\,2} &\sim (n-\tfrac{1}{2})^2\pi^2/\left(\int_o^L \rho^{1/2}dt\right)^2 \end{aligned}\right\} \quad \text{for } n\to\infty$$

it is sufficient to show that

$$F_n \equiv e^{\frac{1}{n}-\ell nn} \prod_{k\neq n} \left(1-\frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_k^{\,2}-\lambda_n^{\,2}}\right) e^{1/k} = O(1) \tag{21a}$$

and

$$G_n \equiv e^{-\frac{1}{n}-\ell nn} \frac{\mu_n^{\,2}}{\mu_1^{\,2}} \prod_{k\neq n} \left(1+\frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\mu_k^{\,2}-\mu_n^{\,2}}\right) e^{-1/k} = O(1) \tag{21b}$$

As a matter of fact, we shall only show that (21a) holds; the same arguments can be used to prove (21b). Since

$$\frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_k^{\,2}-\lambda_n^{\,2}} < 1 \quad \text{for all } k\neq n \quad ,$$

we can bound F_n above and below:

$$\exp\left[\frac{1}{n} - \ell nn + \sum_{k\neq n}\left(\frac{1}{k} - \frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\mu_k^{\,2}-\lambda_n^{\,2}}\right)\right] \le F_n$$

$$\le \exp\left[\frac{1}{n} - \ell nn + \sum_{k\neq n}\left(\frac{1}{k} - \frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_k^{\,2}-\lambda_n^{\,2}}\right)\right] .$$

For large values of n, say n>N, where N denotes the subscript above which $\lambda_n^{\,2}$ and $\mu_n^{\,2}$ take on their asymptotic values, we can rewrite the above inequalities as:

$$\exp\left[\left(\sum_{1}^{n}\frac{1}{k} - \ell nn\right) + \sum_{1}^{N}\frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_n^{\,2}-\mu_k^{\,2}} + \sum_{N+1}^{n-1}\frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_n^{\,2}-\mu_k^{\,2}}\right.$$

$$\left. + \sum_{n+1}^{4n^2}\left(\frac{1}{k} - \frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\mu_k^{\,2}-\lambda_n^{\,2}}\right) + \sum_{4n^2+1}^{\infty}\left(\frac{1}{k} - \frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\mu_k^{\,2}-\lambda_n^{\,2}}\right)\right] \le F_n$$

$$\le \exp\left[\left(\sum_{1}^{n}\frac{1}{k} - \ell nn\right) + \sum_{1}^{N}\frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_n^{\,2}-\lambda_k^{\,2}} + \sum_{N+1}^{n-1}\frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_n^{\,2}-\lambda_k^{\,2}}\right.$$

$$\left. + \sum_{n+1}^{4n^2}\left(\frac{1}{k} - \frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_k^{\,2}-\lambda_n^{\,2}}\right) + \sum_{4n^2+1}^{\infty}\left(\frac{1}{k} - \frac{\lambda_k^{\,2}-\mu_k^{\,2}}{\lambda_k^{\,2}-\lambda_n^{\,2}}\right)\right] .$$

Now

$$\sum_{1}^{N} = 0(n^{-2}) \quad ,$$

$$\sum_{N+1}^{n-1} = \frac{1}{2} \ln n + o(1)$$

$$\sum_{n+1}^{4n^2} = -\frac{1}{2} \ln n + o(1)$$

$$\sum_{4n^2+1}^{\infty} = o(1) \quad ,$$

where we have used the sigma sign as a shorthand notation for the various sums entering in the exponents. Therefore,

$$F_n \to e^{\gamma} \qquad \text{as } n\to\infty \quad ,$$

where γ is Euler's constant.

The next step consists in introducing the sequence space S where

$$S = \left\{ \{\nu_n\}_1^{\infty} : ||\nu|| = \sum_1^{\infty} \frac{|\nu_n|}{n^4} < \infty \right\} , \tag{22}$$

as well as the supspace $S^* \subset S$ of sequences which (i) are positive (ii) are increasing and (iii) satisfy the asymptotic relation

$$\nu_n \sim \frac{n^2\pi^2}{4I^2} \qquad \text{as } n\to\infty \quad , \tag{23}$$

where I is a constant. Clearly, sequences obtained by interlacing two spectra lie in S^*. Also, h can be thought of as a map of S^* in S.

The spaces S and S^* are convex: indeed, if $\sigma, \eta \varepsilon S^*$, then $\alpha\sigma + (1-\alpha)\eta \varepsilon S^*$ where $0 \le \alpha \le 1$. More importantly, the functions $h_n(\alpha\sigma + (1-\alpha)\eta)$, $n=1, 2, \ldots$ are analytic functions of α for $\alpha \varepsilon [0,1]$. Consequently, for every n there exists a value α_n such that

$$h_n(\sigma) - h_n(\eta) = \left.\frac{dh_n}{d\alpha}\right|_{\alpha=\alpha_n}$$

or equivalently

$$h_n(\sigma) - h_n(\eta) = \sum_m \left.\frac{\partial h_n}{\partial \lambda_m^2}\right|_{\alpha_n} (\sigma_{2m} - \eta_{2m})$$

$$+ \sum_m \left.\frac{\partial h_n}{\partial \mu_m^2}\right|_{\alpha_n} (\sigma_{2m-1} - \eta_{2m-1}) \; .$$

We should remark at this stage that on account of the asymptotic form of h_n, the α_n's become independent of n, or more precisely

$$\left.\begin{aligned} \alpha_{2n} &\to \alpha_e \\ \alpha_{2n-1} &\to \alpha_o \end{aligned}\right\} \quad \text{as } n \to \infty \; .$$

Referring to (19), we can deduce explicit formulas for $\partial h_n / \partial \lambda_m^2$ and $\partial h_n / \partial \mu_m^2$. As a result

$$h_{2n}(\sigma)-h_{2n}(\eta) = -\lambda_n^{(2n)^2} f_n^{(2n)} \sum_{m\neq n} \frac{\sigma_{2m} - \eta_{2m}}{\lambda_m^{(2n)^2}(\lambda_m^{(2n)^2}-\lambda_n^{(2n)^2})}$$

$$+ f_n^{(2n)} \left[\frac{1}{\lambda_n^{(2n)^2}} - \sum_k \frac{1}{\mu_k^{(2n)^2}-\lambda_n^{(2n)^2}} + \sum_{k\neq n} \frac{1}{\lambda_k^{(2n)^2}-\lambda_n^{(2n)^2}}\right](\sigma_{2n}-\eta_{2n})$$

$$+ \lambda_n^{(2n)^2} f_n^{(2n)} \sum_m \frac{\sigma_{2m-1} - \eta_{2m-1}}{\mu_m^{(2n)^2}(\mu_m^{(2n)^2}-\lambda_n^{(2n)^2}} \quad , \qquad (24a)$$

where

$$\lambda_m^{(2n)^2} = \alpha_{2n}\,\sigma_{2m} + (1-\alpha_{2n})\eta_{2m} \quad ,$$

$$\mu_m^{(2n)^2} = \alpha_{2n}\,\sigma_{2m-1} + (1-\alpha_{2n})\eta_{2m-1} \quad ,$$

and

$$f_n^{(2n)} = h_{2n}\,(\alpha_{2n}\sigma + (1-\alpha_{2n})\eta) \quad .$$

Similarly

$$h_{2n-1}(\sigma)-h_{2n-1}(\eta) = \mu_n^{(2n-1)^2} g_n^{(2n-1)} \sum_m \frac{\sigma_{2m} - \eta_{2m}}{\lambda_m^{(2n-1)^2}(\lambda_m^{(2n-1)^2}-\mu_n^{(2n-1)^2})}$$

$$-\mu_n^{(2n-1)^2} g_n^{(2n-1)} \sum_{m\neq n} \frac{\sigma_{2m-1} - \eta_{2m-1}}{\mu_m^{(2n-1)^2}(\mu_m^{(2n-1)^2}-\mu_n^{(2n-1)^2})}$$

$$+ 2g_n^{(2n-1)} \sum_m \left\{ \frac{\sigma_{2m}-\eta_{2m}}{\lambda_m^{(2n-1)^2}} - \frac{\sigma_{2m-1} - \eta_{2m-1}}{\mu_m^{(2n-1)^2}} \right\}$$

(24b)

$$+ g_n^{(2n-1)} \left[\frac{2}{\mu_n^{(2n-1)^2}} - \sum_k \frac{1}{\lambda_k^{(2n-1)^2} - \mu_n^{(2n-1)^2}} + \sum_{k\neq n} \frac{1}{\mu_k^{(2n-1)^2} - \mu_n^{(2n-1)^2}} \right] (\sigma_{2n-1} - \eta_{2n-1}) .$$

Because of the asymptotic form of f_n and g_n, we can assume that $h_n(\sigma)-h_n(\eta)$ is positive for $n>2N$, by interchanging if need be σ and η. Dividing (24a) by $(2n)^4$ and (24b) by $(2n-1)^4$ and summing over n, we can see that

$$\left| \sum_1^\infty \frac{h_n(\sigma)-h_n(\eta)}{n^4} \right| \leq \text{const.} \; ||\sigma-\eta|| \; ,$$

or, in view of the previous remark

$$||h(\sigma)-h(\eta)|| - \left\{ \sum_1^{2N} \frac{|h_n(\sigma)-h_n(\eta)|}{n^4} - \sum_L^{2N} \frac{h_n(\sigma)-h_n(\eta)}{n^4} \right\}$$

$$\leq \text{const.} \; ||\sigma-\eta|| \quad . \tag{25}$$

In order to derive the above inequality, we have made use of the fact that the dependence of the various quantities on α_n can be ignored for large n's and that the sums

$$\sum_{n\neq m} \frac{1}{\lambda_n^2-\lambda_m^2} \, , \; \sum_n \frac{1}{\lambda_n^2-\mu_m^2} \, , \; \sum_n \frac{1}{\mu_n^2-\lambda_m^2} \, , \; \sum_{n\neq m} \frac{1}{\mu_n^2-\mu_m^2}$$

are of $O(m^{-2})$. Noting that (24a) and (24b) imply that

$$|h_n(\sigma)-h_n(\eta)| \leq \text{const.}\ ||\sigma-\eta|| \tag{26}$$

we therefore conclude that

$$||h(\sigma) - h(\eta)|| \leq \text{const.}\ ||\sigma-\eta|| \ . \tag{27}$$

This Lipschitz condition guarantees the existence of a unique solution to the initial value problem.

4. The Quasi-homogeneous String

This paragraph examines what happens to formula (13) for the case of a quasi-homogeneous string, i.e. for a string whose density is:

$$\rho = \bar{\rho}\ [1 + \varepsilon r(x)] \tag{28}$$

where $\varepsilon \ll 1$. Making use of perturbation theory, one can easily show that

$$\lambda_n^{\ 2}(\ell) = \frac{n^2\pi^2}{\bar{\rho}(L-\ell)^2}\,[1 + \varepsilon(-r_o + \tfrac{1}{2}\, r_{2n}) + O(\varepsilon^2)] \ , \tag{28a}$$

and

$$\mu_n^{\ 2}(\ell) = \frac{(n-\frac{1}{2})^2\pi^2}{\bar{\rho}(L-\ell)^2}\,[1 - \varepsilon(r_o + \tfrac{1}{2}\, r_{2n-1}) + O(\varepsilon^2)] \ , \tag{28b}$$

where

$$r_o = \frac{1}{L-\ell}\int_\ell^L r(x)dx \ , \tag{29}$$

and

$$r_n = \frac{2}{L-\ell} \int_{\ell}^{L} r(x) \cos n\pi \frac{x-\ell}{L-\ell} \, dx \quad . \tag{30}$$

Therefore

$$\frac{\lambda_n^{4}(\ell)}{\mu_{n+1}^{2}(\ell)\mu_n^{2}(\ell)} = \frac{n^4}{(n^2-\frac{1}{4})^2} [1 + \varepsilon(r_{2n} + \frac{1}{2} r_{2n+1} + \frac{1}{2} r_{2n-1}) + 0(\varepsilon^2)]$$

and consequently

$$\prod_{n=1}^{\infty} \frac{\lambda_n^{4}(\ell)}{\mu_{n+1}^{2}(\ell)\mu_n^{2}(\ell)} = \prod_{n=1}^{\infty} \frac{n^4}{(n^2-\frac{1}{4})^2} [1 + \varepsilon \sum_{n=1}^{\infty} (r_{2n} + \frac{r_{2n+1}+r_{2n-1}}{2}) + 0(\varepsilon^2)]$$

$$= \frac{\pi^2}{4} \{1 + \varepsilon(-\frac{r_1}{2} + \sum_{n=1}^{\infty} r_n) + 0(\varepsilon^2)\} \ .$$

Hence

$$\frac{1}{(L-\ell)^2\mu_1^{2}(\ell)} \prod_{n=1}^{\infty} \frac{\lambda_n^{4}(\ell)}{\mu_{n+1}^{2}(\ell)\ {}_n^{2}(\ell)} = \bar{\rho}\ \frac{1 + \varepsilon(-\frac{r_1}{2} + \sum_1^{\infty} r_n) + 0(\varepsilon^2)}{1 - \varepsilon(r_o + \frac{r_1}{2}) + 0(\varepsilon^2)}$$

$$= \bar{\rho}\ [1 + \varepsilon \sum_{o}^{\infty} r_n + 0(\varepsilon^2)] \tag{31}$$

Thus, for the simple case of the quasi-homogeneous string, formula (13) reduces to an end-point evaluation of the Fourier cosine series for $r(x)$. Indeed, from the representation

$$r(x) = r_o + \sum_1^\infty r_n \cos n\pi \frac{x-\ell}{\ell-\ell}$$

it is obvious that

$$\rho(\ell) = \bar{\rho}\ [1 + \varepsilon \sum_o^\infty r_n]\ .$$

5. Concluding Remarks

The derivation of $\rho(0)$ via the computation of ℓ_o and m_1 can be thought of as involving a process in which the frequency ω tends to infinity. Had we started with a time dependent formulation of the problem, this process would have been associated with letting the time t tend to zero. In that sense, the method is reminiscent of that of Kac [3], Deift & Trubowitz [4] and Trubowitz [5]. A similar approach can also be used for solving the canonical Sturm-Liouville problem: in this case q(x) is expressed via a trace formula such as the ones given in [6,7,8]. However, the results are not as explicit as in the present case.

Finally, we have restricted our analysis to the boundary conditions (2) and (3). If these boundary conditions are replaced by the following ones:

$$u(0) = u(L)\ \cos\gamma + u'(L)\sin\gamma = 0 \qquad (1')$$

and

$$u(0)\ \cos\beta - u'(0)\ \sin\beta = u(L)\ \cos\gamma + u'(L)\ \sin\gamma = 0 \qquad (2')$$

where $0 < \beta \leq \pi/2$ and $0 \leq \gamma < \pi/2$, then the expressions for ℓ_o and m_1 change slightly (see [9]) and as a result formula (13) becomes:

$$\rho(x) = \frac{1}{\sin^2\beta\ (L-x)^2\ \mu_1{}^2(x)} \prod_{n=1}^{\infty} \frac{\lambda_n{}^4(x)}{\mu_{n+1}^2(x)\mu_n{}^2(x)}\ . \qquad (13')$$

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References

[1] Krein, M. G., On inverse problems for an inhomogeneous string, (Russian), Dokl. Akad. Nauk. SSSR, vol. 82, 1952, pp. 669-672.

[2] Krein, M. G., Some new problems in the theory of Sturmian systems, (Russian), Prikh. Matem. Mekh., vol. 16, 1952, pp. 555-568.

[3] Kac, M., Can one hear the shape of a drum? Amer. Math. Monthly, vol. 73, 1966, pp. 1-23.

[4] Deift, P. & Trubowitz, E., Inverse scattering on the line, Comm. Pure Appl. Math., vol. 32, 1979, pp. 121-251.

[5] Trubowitz, E., The inverse problem for periodic potentials, Comm. Pure Appl. Math., vol. 30, 1977, pp. 321-337.

[6] Gelfand, I. M. & Levitan, B. M., On a simple identity for the eigenvalues of a differential operator of the second order (Russian), Dokl. Akad. Nauk. SSSR, vol. 88, 1953, pp. 593-596.

[7] Halberg Jr., C. J. A. & Kramer, V. A., A generalization of the trace concept, Duke Math. J., vol. 27, 1960, pp. 607-617.

[8] Barcilon, V., A note on a formula of Gelfand and Levitan, J. Math. Anal. Appl., vol. 48, 1974, pp. 43-50.

[9] Barcilon, V. & Turchetti, G., Extremal solutions of inverse eigenvalue problems with finite spectral data, Wave Motion, vol. 2, 1980, pp. 139-148.

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